Entransy—A physical quantity describing heat transfer ability

Zeng-Yuan Guo [*], Hong-Ye Zhu, Xin-Gang Liang

Department of Engineering Mechanics, The Beijing Key Heat Transfer and Energy Utilization Lab, Tsinghua University, Beijing 100084, China

Abstract

A new physical quantity, $E_h = \frac{1}{2} Q_{vh} T$, has been identified as a basis for optimizing heat transfer processes in terms of the analogy between heat and electrical conduction. This quantity, which will be referred to as entransy, corresponds to the electric energy stored in a capacitor. Heat transfer analyses show that the entransy of an object describes its heat transfer ability, as the electrical energy in a capacitor describes its charge transfer ability. Entransy dissipation occurs during heat transfer processes as a measure of the heat transfer irreversibility. The concepts of entransy and entransy dissipation were used to develop the extremum principle of entransy dissipation for heat transfer optimization. For a fixed boundary heat flux, the conduction process is optimized when the entransy dissipation is minimized, while for a fixed boundary temperature the conduction is optimized when the entransy dissipation is maximized. An equivalent thermal resistance for multi-dimensional conduction problems is defined based on the entransy dissipation, so that the extremum principle of entransy dissipation can be related to the minimum thermal resistance principle to optimize conduction. For examples, the optimum thermal conductivity distribution was obtained based on the extremum principle of entransy dissipation for the volume-to-point conduction problem. The domain temperature is substantially reduced relative to the uniform conductivity case. Finally, a brief introduction on the application of the extremum principle of entransy dissipation to heat convection is also provided.
© 2007 Elsevier Ltd. All rights reserved.

Keywords: Entransy; Dissipation; Heat transfer optimization; Extremum principle of entransy dissipation

1. Introduction

Designers are always seeking ways to improve heat transport techniques in many engineering fields because they can improve the energy utilization efficiency or reduce the weight and size of the heat transfer equipment. For instance, adding high heat conductivity materials to the basis materials can increase the thermal conduction rate and increasing the fluid velocity or turbulence intensity can enhance the convection heat transfer rate. There are various quantities to describe the heat transfer rate, but there is no concept of efficiency for transfer processes because in heat transfer problems the input (for example, high conductivity materials or fluid velocity) has different units than the output (increased heat transfer rate or

reduced temperature difference). As a result, a heat transfer process can be enhanced, but there is no way to know how to optimize a heat transfer process.

Heat transfer is an irreversible, non-equilibrium process from the point of view of thermodynamics. Onsager [1,2] set up the fundamental equations for non-equilibrium thermodynamic processes and derived the principle of the least dissipation of energy using variational theory. Prigogine [3] developed the principle of minimum entropy production based on the idea that the entropy production of a thermal system at steady-state should be the minimum. The integral expression for the minimum entropy production principle can be used to derive the partial differential equations for heat conduction, mass diffusion, viscous flow, etc. However, both of these principles do not deal with heat transfer optimization. Bejan [4,5] developed entropy generation expressions for heat and fluid flows. He analyzed the least combined entropy production induced by the heat transfer and the fluid viscosity as the objective function to optimize

---

[*] Corresponding author. Tel.: +86 10 6278 2660; fax: +86 10 6278 3771.
E-mail address: demgzy@tsinghua.edu.cn (Z.-Y. Guo).

Nomenclature

A	surface area	$\dot{Q}_h$	heat flow (thermal current)
C_e	capacitance	$\dot{q}_{hs}$	heat source intensity
C_h	heat capacity (thermal capacitance)	Q_{ve}	electrical charge in a capacitor
c_v	specific heat at constant volume	Q_{vh}	heat stored in an object (thermal charge)
c_p	specific heat at constant pressure	R_e	electrical resistance
E_e	electrical energy	R_h	thermal resistance
$\dot{E}_e$	electrical energy flow	t	time
$\dot{E}_{e\phi}$	dissipation rate of electrical energy	T	temperature
E_h	entransy	U_e	electrical potential
$\dot{E}_h$	entransy flow	$U_h = T$	thermal potential (temperature)
$E_{h\phi}$	entransy dissipation	V	volume
$\dot{E}_{h\phi}$	dissipation rate of entransy		
E_{ve}	electrical energy in a capacitor	*Greek symbols*	
$\dot{E}_{ve}$	derivative of E_{ve} with respect to time	δ	thickness, variation
E_{vh}	entransy stored in an object	η	entransy transfer efficiency
$\dot{E}_{vh}$	derivative of E_{vh} with respect to time	ε	convergence criteria
h	convection coefficient	ε_h	entransy density
J	functional	$\dot{\varepsilon}_h$	entransy flux
k	thermal conductivity	λ	Lagrange multiplier
M	mass	ρ	density
Q_e	electrical charge	μ	dynamic viscosity
$\dot{Q}_e = I$	electric charge flow (electric current)	ϕ	mechanical energy dissipation function
Q_h	heat	ϕ_h	entransy dissipation function
$\dot{q}_h$	heat flux	ϕ_e	electrical energy dissipation function

the geometry of heat transfer tubes and to find optimized parameters for heat exchangers and thermal systems. This type of investigation is called thermodynamic optimization because its objective is to minimize the total entropy generation due to flow and thermal resistance. For the volume-to-point heat conduction problem, Bejan [6,7] developed a constructal theory network of conducting paths that determines the optimal distribution of a fixed amount of high conductivity material in a given volume such that the overall volume-to-point resistance is minimized. This article introduces a new physical quantity, entransy, which can be used to define the efficiencies of heat transfer processes and to optimize heat transfer processes. The entransy corresponds to the electrical potential energy in a capacitor, and is an indication of both the nature of "energy" and the heat transfer ability.

2. Analogy between heat and electrical conduction

Experimental studies often used the electric conduction analogy to heat conduction to solve complex steady-state or transient heat conduction problems [8] in the 1950s because computers were not well developed and thermal experiments were cumbersome. The two systems are analogous because Fourier's law for heat conduction is analogous to Ohm's law for electrical circuits. In the analogy, the heat flow corresponds to the electrical current, the thermal resistance to the electrical resistance, temperature to

electric voltage, and heat capacity to capacitance. The analogies between the parameters for the two processes are listed in Table 1 from which shows that the thermal system lacks the parameter corresponding to the electrical potential energy of a capacitor.

An appropriate quantity, E_{vh}, can be defined for an object that corresponds to the electrical energy in a capacitor based on the often used analogy between electrical and thermal systems. The quantity E_{vh} is defined as:

$$E_{vh} = \frac{1}{2} Q_{vh} U_h = \frac{1}{2} Q_{vh} T \tag{1}$$

where $Q_{vh} = M c_v T$ is the thermal energy or the heat stored in an object with constant volume which may be referred to as the thermal charge. U_h or T represents the thermal potential. The next section further discusses the physical meaning of this quantity.

3. Entransy

The physical meaning of entransy can be understood by considering a reversible heating process of an object with temperature, T, and specific heat at constant volume, c_v. For a reversible process, the temperature difference between the object and the heat source and the heat added are infinitesimal, as shown in Fig. 1. Continuous heating of the object implies an infinite number of heat sources that heat the object in turn. The temperature of these heat

Table 1
Analogies between electrical and thermal parameters

| Electrical charge stored in capacitor Q_{ve} [C]	Electrical current (charge flux) I [C]/[s] = [A]	Electrical resistance R_e [Ω]	Capacitance $C_e = Q_{ve}/U_e$ [F]
Heat stored in a body $Q_{vh} = Mc_vT$ [J]	Heat flow $\dot{Q}_h$ [J/s]	Thermal resistance R_h [s K/J]	Heat capacity $C_h = Q_{vh}/T$ [J/K]
Electrical potential U_e [V]	Electrical current density $\dot{q}_e$ [C/m² s]	Ohm's law $\dot{q}_e = -K_e \frac{dU_e}{dn}$	Electrical potential energy in a capacitor $E_e = \frac{1}{2}Q_e U_e$ [J]
Thermal potential (temperature) $U_h = T$ [K]	Heat flux density $\dot{q}_h$ [J/m² s]	Fourier law $\dot{q}_h = -K_h \frac{dU_h}{dn}$	?
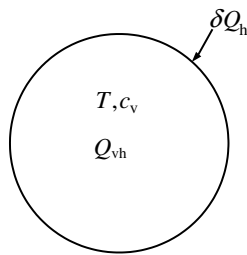
Fig. 1. Spheric thermal capacitor.

sources increases infinitesimally with each source giving an infinitesimal amount of heat to the object. The temperature represents the potential of the heat because the availability of the heat differs at different temperatures. Hence the "potential energy" of the thermal energy increases in parallel with the increasing thermal energy (thermal charge) when heat is added. When an infinitesimal amount of heat is added to an object, as with the derivation for the electrical energy in a capacitor, the increment in "potential energy" of the thermal energy can be written as the product of the thermal charge and the thermal potential (temperature) differential

$$dE_{vh} = Q_{vh}dT \tag{2a}$$

If absolute zero is taken as the zero temperature potential, then the "potential energy" of the thermal energy in the object at temperature T is

$$E_{vh} = \int_0^T Q_{vh}\,dT \tag{2b}$$

The word potential energy is quoted because its unit is J K, not joules. For a constant specific heat

$$E_{vh} = \int_0^T Mc_v T\,dT = \frac{1}{2}Mc_v T^2 \tag{2c}$$

Hence, like an electric capacitor which stores electric charge and the resulting electric potential energy, an object can be regarded as a thermal capacitor which stores heat (thermal charge) and the resulting thermal "potential energy".

If the object is put in contact with an infinite number of heat sinks that have infinitesimally lower temperatures, the total quantity of "potential energy" of heat which can be transferred out is $\frac{1}{2}Q_{vh}T$. Hence the "potential energy" represents the heat transfer ability of an object. This concept is called *Entransy* because it possesses both the nature of "energy" and the transfer ability. This has also been referred to as the heat transport potential capacity in an earlier paper by the authors [9].

The concept of entransy was derived here in terms of the analogy between electrical conduction and heat conduction for the heating of an object. Biot [10] introduced a similar concept in the 1950s in his derivation of the differential conduction equation using the variation method. Eckert and Drake [11] summarized that Biot's formulation of a variational equivalent of the thermal conduction equation from the ideas of irreversible thermodynamics to define a thermal potential and a variational invariant. The thermal potential plays a role analogous to the potential energy, while the variational invariant is related to the concept of dissipation function. However, Biot did not further expand on the physical meaning of the thermal potential and its application to heat transfer optimization was not found later except in approximate solutions to anisotropic conduction problems.

4. Entransy dissipation and balance equation

The concept of entransy dissipation will again be analyzed by analogy between electric conduction and heat conduction. Fig. 2 shows a typical electrical system with two capacitors and a resistor. The charge and potential on capacitor 1 before charging are Q_{ve10} and U_{ve10}, while those
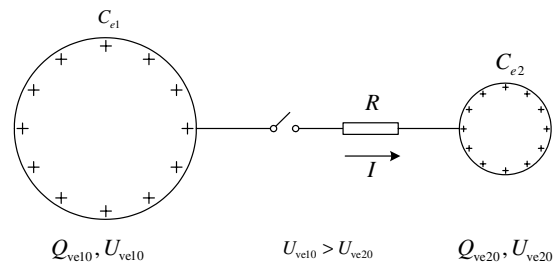

Fig. 2. Schematic of the charging/discharging process between two capacitors.

on capacitor 2 are Q_{ve20} and U_{ve20}. With Q_{ve} representing the charge on the capacitor, Q_e the charge flowing through the resistor and I the current at any time after the switch has been closed, the charge conservation equation is:

$$I = \dot{Q}_e = \dot{Q}_{ve1} = \frac{dQ_{ve1}}{dt} = \dot{Q}_{ve2} = \frac{dQ_{ve2}}{dt} \qquad (3)$$

During the discharging/charging process, the decrease in electric energy in capacitor 1 is equal to the increase in electric energy in capacitor 2 plus the dissipation of electrical energy in the resistor. So, electrical energy balance gives

$$Q_{ve1}\frac{dU_{e1}}{dt} + Q_{ve2}\frac{dU_{e2}}{dt} + I^2R = 0 \qquad (4)$$

Eqs. (3) and (4) can be solved to determine the electrical system parameters.

Charges on capacitors 1 and 2

$$Q_{ve1} = B_{e1}^* + C_e^*(U_{e10} - U_{e20})e^{-t/R_eC_e^*} \qquad (5a)$$

$$Q_{ve2} = B_{e2}^* - C_e^*(U_{e10} - U_{e20})e^{-t/R_eC_e^*} \qquad (5b)$$

Current

$$I = \dot{Q}_e = \frac{dQ_e}{dt} = \left(\frac{U_{e10} - U_{e20}}{R_e}\right)e^{-t/R_eC_e^*} \qquad (5c)$$

Potentials of capacitors 1 and 2

$$U_{e1} = \frac{B_{e1}^*}{C_{e1}} + \frac{C_e^*}{C_{e1}}(U_{e10} - U_{e20})e^{-t/R_eC_e^*} \qquad (5d)$$

$$U_{e2} = \frac{B_{e2}^*}{C_{e2}} - \frac{C_e^*}{C_{e2}}(U_{e10} - U_{e20})e^{-t/R_eC_e^*} \qquad (5e)$$

Electric energy dissipation rate

$$\dot{E}_{e\phi} = I^2 R_e = \frac{(U_{e10} - U_{e20})^2}{R_e}e^{-2t/R_eC_e^*} \qquad (5f)$$

Rate of electric energy lost by capacitor 1 and received by capacitor 2

$$\dot{E}_{e1} = \dot{E}_{ve1} = \frac{(U_{e10} - U_{e20})}{R_e}$$
$$\times \left[\frac{B_{e1}^*}{C_{e1}}e^{-t/R_eC_e^*} + \frac{C_e^*}{C_{e1}}(U_{e10} - U_{e20})e^{-2t/R_eC_e^*}\right] \qquad (5g)$$

$$\dot{E}_{e2} = \dot{E}_{ve2} = \frac{(U_{e10} - U_{e20})}{R_e}$$
$$\times \left[\frac{B_{e2}^*}{C_{e2}}e^{-t/R_eC_e^*} - \frac{C_e^*}{C_{e2}}(U_{e10} - U_{e20})e^{-2t/R_eC_e^*}\right] \qquad (5h)$$

where

$$C_e^* = \frac{C_{e1}C_{e2}}{C_{e1} + C_{e2}}, \quad B_1^* = \frac{C_{e1}(Q_{ve10} + Q_{ve20})}{C_{e1} + C_{e2}},$$

$$B_2^* = \frac{C_{e2}(Q_{ve10} + Q_{ve20})}{C_{e1} + C_{e2}}, \quad \dot{E}_{e1} = U_{e1}I, \quad \dot{E}_{e2} = U_{e2}I,$$

$$\dot{E}_{ve1} = \frac{dE_{ve1}}{dt}, \quad \dot{E}_{ve2} = \frac{dE_{ve2}}{dt}.$$

Eqs. (5b) and (5g) show that the charging time for the charge on capacitor 2 to increase from Q_{ve20} to Q_{ve2}

depends not only on the potential of capacitor 1, but also on its capacitance, that is, the larger the capacitance of capacitor 1 the shorter the charging time is. Hence, the charging time of capacitor 2 will decrease as the electric energy stored in capacitor 1 increases. This implies that the electric energy describes the ability to charge other capacitors or the ability to transfer charge.

The ratio of the electric energy received by capacitor 2 to the electric energy delivered by capacitor 1 can be defined as the electric energy transfer efficiency

$$\eta_e = \frac{U_{e2}I}{U_{e1}I} = 1 - \frac{I^2R_e}{U_{e1}I} = 1 - \frac{IR_e}{U_{e1}} \qquad (5i)$$

which shows that the electric resistance determines the electric energy transfer efficiency for given I and U_{e1}.

Now, consider the thermal system shown in Fig. 3, which is composed of objects 1 and 2 which have very high thermal conductivities and thermal capacitances and object 3 which has very low thermal conductivity and thermal capacitance. Therefore, objects 1 and 2 can be regarded as thermal capacitor without thermal resistance while object 3 is a thermal resistor without thermal capacitance.

The thermal charge and potential (temperature) for thermal capacitors 1 and 2 before they touch are Q_{vh10}, Q_{vh20}, $U_{h10} = T_{10}$, and $U_{h20} = T_{20}$. Let Q_{vh} represents the thermal charge in the capacitor, Q_h the heat going through the thermal resistor from initial state to the final state, and $\frac{dQ_h}{dt}$ the heat flow through the resistor after the three objects touch. While energy flows from the high temperature object 1 to the low temperature object 2, conservation of thermal energy (or thermal charge) gives

$$\dot{Q}_h = \frac{dQ_h}{dt} = \frac{dQ_{vh1}}{dt} = \frac{dQ_{vh2}}{dt} \qquad (6)$$
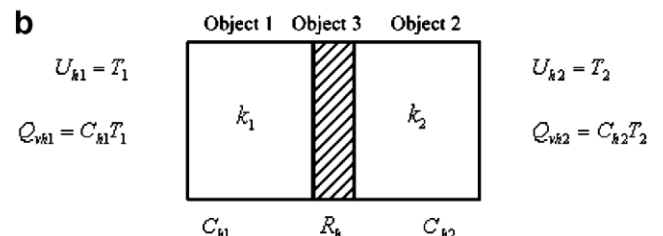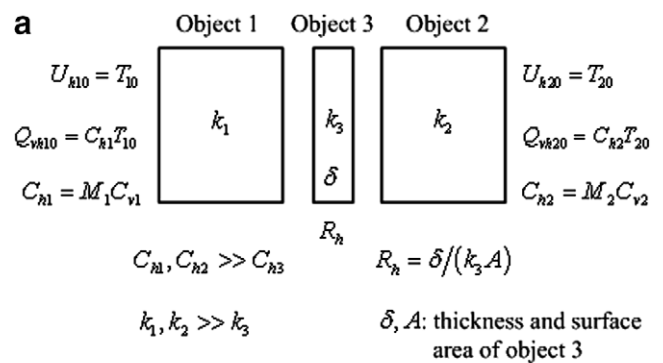

Fig. 3. (a) Thermal system before contact and (b) thermal system after contact.

Meanwhile there is also an accompanying flow of entransy from object 1 to object 2 in this thermal system. However, entransy, unlike the thermal energy, is not conserved during the heat transfer process because of entransy dissipation in the thermal resistor. Hence, the entransy decrease of thermal capacitor 1 is equal to the entransy increase of thermal capacitor 2 plus its dissipation in the thermal resistor

$$Q_{vh1}\frac{dT_1}{dt} + Q_{vh2}\frac{dT_2}{dt} + \dot{Q}_h^2 R_h = 0 \tag{7}$$

where $\dot{Q}_h^2 R_h$ is the entransy dissipation rate in a thermal resistor, which is analogous to the electric energy dissipation rate in a electric resistor. Eq. (7) is then an entransy balance equation. The solution of Eqs. (6) and (7) gives the instantaneous parameters of the thermal system. The thermal energy stored in object 1 and object 2

$$Q_{vh1} = B_{h1}^* + C_h^*(T_{10} - T_{20})e^{-t/R_h C_h^*} \tag{8a}$$

$$Q_{vh2} = B_{h2}^* - C_h^*(T_{10} - T_{20})e^{-t/R_h C_h^*} \tag{8b}$$

Heat flow

$$\dot{Q}_h = \frac{dQ_h}{dt} = \left(\frac{T_{10} - T_{20}}{R_h}\right)e^{-t/R_h C_h^*} \tag{8c}$$

Temperature (thermal potential) of object 1 and object 2

$$T_1 = \frac{B_{h1}^*}{C_{h1}} + \frac{C_h^*}{C_{h1}}(T_{10} - T_{20})e^{-t/R_h C_h^*} \tag{8d}$$

$$T_2 = \frac{B_{h2}^*}{C_{h2}} + \frac{C_h^*}{C_{h2}}(T_{10} - T_{20})e^{-t/R_h C_h^*} \tag{8e}$$

Rate of entransy delivered by object 1 and received by object 2

$$\dot{E}_{h1} = \dot{E}_{vh1} = \frac{(T_{10} - T_{20})}{R_h} \\ \times \left[\frac{B_{h1}^*}{C_{h1}}e^{-t/R_h C_h^*} + \frac{C_h^*}{C_{h1}}(T_{10} - T_{20})e^{-2t/R_h C_h^*}\right] \tag{8f}$$

$$\dot{E}_{h2} = \dot{E}_{vh2} = \frac{(T_{10} - T_{20})}{R_h} \\ \times \left[\frac{B_{h2}^*}{C_{h2}}e^{-t/R_h C_h^*} - \frac{C_h^*}{C_{h2}}(T_{10} - T_{20})e^{-2t/R_h C_h^*}\right] \tag{8g}$$

Rate of dissipation of entransy in the thermal resistor (object 3)

$$\dot{E}_{h\phi} = \dot{Q}_h^2 R_h \tag{8h}$$

where

$$C_h^* = \frac{C_{h1}C_{h2}}{C_{h1} + C_{h2}}, \quad B_1^* = \frac{C_{h1}(Q_{vh10} + Q_{vh20})}{C_{h1} + C_{h2}},$$

$$B_2^* = \frac{C_{h2}(Q_{vh10} + Q_{vh20})}{C_{h1} + C_{h2}}, \quad \dot{E}_{h1} = T_1\dot{Q}_h = T_1\frac{dQ_{vh1}}{dt},$$

$$\dot{E}_{h2} = T_2\dot{Q}_h = T_2\frac{dQ_{vh2}}{dt}.$$

Eqs. (8b) and (8f) show that the heating time to change the thermal charge on object 2 from Q_{vh20} to Q_{vh2} depends not only on the temperature, but also on the heat capacity of object 1. Hence, the larger entransy stored in object 1 results in less heating time for object 2. Thus the entransy describes the ability of an object to transfer heat just as the electrical energy describes the ability of a capacitor to transfer charge. Since a certain amount of entransy is dissipated in the thermal resistor, an entransy transfer efficiency can be defined as the ratio of the entransy flow into object 2 to the entransy flow out of object 1

$$\eta_h = \frac{T_2\dot{Q}_h}{T_1\dot{Q}_h} = 1 - \frac{R_h\dot{Q}_h^2}{T_1\dot{Q}_h} = 1 - \frac{R_h\dot{Q}_h}{T_1} \tag{9}$$

A smaller thermal resistance in the thermal system for a given heat flow, $\dot{Q}_h$, results in a higher entransy transfer rate.

Therefore, as an irreversible process, heat transfer through a medium is just like fluid flow through a pipe where mechanical energy is dissipated due to flow friction or electricity flow through a conductor where electrical energy is dissipated due to the electrical resistance. Entransy is also dissipated due to the thermal resistance when heat is conducted though a medium. The thermal resistance originates from the entransy dissipation just as the flow resistance originates from mechanical energy dissipation.

Since the internal thermal resistance of an object in general is not negligible, an entransy balance equation for a continuum can also be derived. For heat conduction without a heat source, the thermal energy conservation equation is

$$\rho c_v\frac{\partial T}{\partial t} = -\nabla \cdot \dot{\mathbf{q}} = \nabla \cdot (k\nabla T) \tag{10a}$$

The entransy balance equation is

$$T\rho c_v\frac{\partial T}{\partial t} = -\nabla \cdot (\dot{\mathbf{q}}T) + \dot{\mathbf{q}} \cdot \nabla T \tag{10b}$$

The left term in Eq. (10b) is the time variation of the entransy stored per unit volume. The first term on the right is the entransy transfer associated with the heat transfer while the second term on the right is the local rate of entransy dissipation. The entransy balance equation can then be rewritten as

$$\frac{d\varepsilon_{vh}}{dt} = -\nabla \cdot (\dot{\varepsilon}_h) - \phi_h \tag{10c}$$

where ε_{vh} is the entransy density, the entransy per unit volume, and $\dot{\varepsilon}_h$ is the entransy flux. The last term on the right side of Eq. (10c) is the entransy dissipation function

$$\phi_h = -\dot{\mathbf{q}} \cdot \nabla T = k(\nabla T)^2 \tag{11}$$

where k is the thermal conductivity and ∇T is the temperature gradient. The physical meaning is the dissipation of entransy per unit time and per unit volume. The entransy dissipation function resembles the dissipation function for mechanical energy in fluid flow.
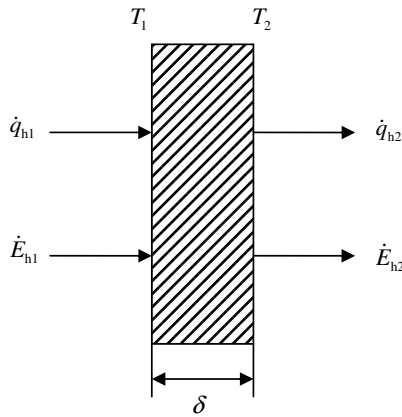
Fig. 4. Steady heat conduction.

Consider one-dimensional steady-state heat conduction in a plate with thickness δ as shown in Fig. 4 where the input heat flux is equal to the output heat flux,

$$\dot{q}_{h1} = \dot{q}_{h2} = \dot{q}_h \tag{12a}$$

However, the input entransy flux is not equal to the output entransy flux due to dissipation during the heat transport. The entransy balance equation is

$$\dot{q}_{h1}T_1 = \dot{q}_{h2}T_2 + \int_0^\delta \phi_h \, dx \tag{12b}$$

where

$$\int_0^\delta \phi_h \, dx = - \int_0^\delta \dot{q}_h \frac{dT}{dx} \, dx = \dot{q}_h \int_{T_2}^{T_1} dT = \dot{q}_h (T_1 - T_2)$$

Eq. (12b) again shows that the input entransy flux is equal to the sum of the output entransy flux and the dissipated entransy per unit time and per unit volume. The entransy transfer efficiency is then

$$\eta_h = \frac{\dot{E}_{h1} - \dot{E}_{h\phi}}{\dot{E}_{h1}} = \frac{\dot{E}_{h2}}{\dot{E}_{h1}} = \frac{T_2}{T_1} \tag{13}$$

which is the same as Eq. (9).

Next consider the transient heat conduction in a continuum. Consider two cubic objects with the same volume, mass, specific heat and thermal conductivity, as shown in Fig. 5. Their initial uniform temperatures are T_1 and T_2 with $T_1 > T_2$. When the two objects touch heat will flow from the high temperature object to the lower temperature
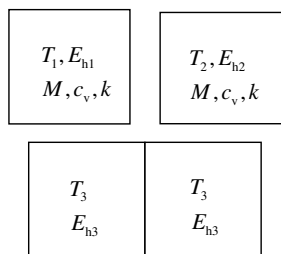

Fig. 5. Transient heat conduction.

object. The difference between this case and that shown in Fig. 3 is that the objects in Fig. 5 have distributed thermal resistances and heat capacities. The internal resistances of the two objects induce entransy dissipation during the heat transfer process. The two objects will come to an equilibrium temperature, T_3, after a sufficiently long time,

$$T_3 = \frac{T_1 + T_2}{2} \tag{14}$$

Before touching, the entransies in the two objects are

$$E_{vh1} = \frac{1}{2}Q_{vh1}T_1 = \frac{1}{2}Mc_vT_1^2$$
$$E_{vh2} = \frac{1}{2}Q_{vh2}T_2 = \frac{1}{2}Mc_vT_2^2 \tag{15a}$$

After touching and equilibrium is achieved, the entransy of each object is

$$E_{vh3} = \frac{1}{2}Q_{vh3}T_3 = \frac{1}{2}Mc_vT_3^2 \tag{15b}$$

Comparison of Eqs. (14), (15a) and (15b) shows that the total entransy of the system decreases after touching. The entransy balance equation is then

$$E_{vh1} + E_{vh2} = 2E_{vh3} + E_{h\phi} \tag{16}$$

where

$$E_{h\phi} = \frac{1}{4}Mc_v(T_1 - T_2)^2 \tag{17}$$

is the dissipated entransy which can also be expressed by integrating the dissipation function over the volumes

$$E_{h\phi} = \int_0^\infty \int_{V_1} k(\nabla T)^2 \, dV \, dt + \int_0^\infty \int_{V_2} k(\nabla T)^2 \, dV \, dt \tag{18}$$

The entransy transfer efficiency from the initial state to the equilibrium state is

$$\eta = \frac{2E_{vh3}}{E_{vh1} + E_{vh2}} = \frac{2T_3^2}{T_1^2 + T_2^2} = \frac{(T_1 + T_2)^2}{2(T_1^2 + T_2^2)} \tag{19}$$

For both the steady-state and transient cases the entransy transfer efficiency does not depend on the thermal conductivity as can be seen from Eqs. (13) and (19). However, larger temperature differences between the two objects result in more dissipation and lower entransy transfer efficiency. Less time is needed for two objects with higher thermal conductivities to reach equilibrium when the entransy dissipation per unit time is larger. Hence, the entransy transfer efficiency between the two objects is not a function of the thermal conductivity.

5. Extremum principle of entransy dissipation and the minimum thermal resistance principle

5.1. Extremum principle of entransy dissipation

The concept of transfer efficiency and heat transfer optimization have not developed because the 'input' and 'output' for heat transfer processes do not have the same

physical parameters. However, the entransy and entransy dissipation concepts provide a mechanism for the optimization of heat transfer processes.

For simplicity consider the optimization of a steady-state heat conduction problem. Cheng et al. [12,13] started from the differential form of the conduction equation to derive a variational statement of the heat conduction using the method of weighted residuals. They derived a minimum entransy dissipation principle for prescribed heat flux boundary conditions and a maximum entransy dissipation principle for prescribed temperature boundary conditions that are referred to as the extremum principle of entransy dissipation.

The least entransy dissipation principle states that for the prescribed heat flux boundary conditions, the least entransy dissipation in the domain leads to the minimum difference between the two boundary temperatures. The principle can be expressed as

$$\dot{Q}_{\mathrm{h}}\delta(\Delta T) = \delta \int_V \frac{1}{2}k(\nabla T)^2 \,\mathrm{d}V = 0 \qquad (20)$$

where δ denotes the variation, ΔT is the temperature difference, and $\dot{Q}_{\mathrm{h}}$ is the heat flow. The maximum entransy dissipation principle states that the largest entransy dissipation in a domain with a prescribed temperature difference as the boundary condition leads to the maximum heat flux. The principle can be expressed as

$$\Delta T \delta \dot{Q}_{\mathrm{h}} = \delta \int_V \frac{1}{2}k(\nabla T)^2 \,\mathrm{d}V = 0 \qquad (21)$$

Unlike the Biot's variational method, the present method works directly with the differential equation and boundary conditions. Furthermore, Biot's variational principle is a quasi-variational principle, as Finlayson [14,15] indicated, which applies only to the approximate solution, not necessarily to the exact solution of heat conduction. He also showed that the Euler equation developed from the theorem of minimum entropy production reduces to the heat conduction equation only when $kT^2 = $ constant.

5.2. Minimum thermal resistance principle

The thermal resistance is defined as the ratio of the temperature difference to the heat flux. This thermal resistance definition, as well as the electrical resistance definition, is only valid for one-dimensional problems. The thermal resistance for multi-dimensional heat transfer problems is difficult to define, especially with non-isothermal boundary conditions. However, an equivalent thermal resistance can be defined for multi-dimensional problems with complex boundary conditions based on the concept of entransy dissipation.

Consider one-dimensional electrical conduction, where the electrical energy dissipation per unit time is

$$\dot{E}_{\mathrm{ve}\phi} = \int_V \phi_{\mathrm{e}} \,\mathrm{d}V = I^2 R_{\mathrm{e}} = \frac{(\Delta U)^2}{R_{\mathrm{e}}} \qquad (22a)$$

where ΔU is the potential difference across the resistance R_{e}. Thus

$$R_{\mathrm{e}} = \frac{\dot{E}_{\mathrm{ve}\phi}}{I^2} = \frac{(\Delta U)^2}{\dot{E}_{\mathrm{ve}\phi}} \qquad (22b)$$

For a multi-dimensional domain with two iso- or non-isopotential boundary conditions, the equivalent electrical resistance can be defined as

$$R_{\mathrm{e}} = \frac{(\Delta U)^2}{\dot{E}_{\mathrm{ve}\phi}} \quad \text{and} \quad R_{\mathrm{e}} = \frac{(\overline{\Delta U})^2}{\dot{E}_{\mathrm{ve}\phi}} \qquad (23a)$$

that is, the equivalent resistance is equal to the potential difference squared divided by the electrical energy dissipation, where $\overline{\Delta U}$ is the mean potential difference. For a multi-dimensional electrical conduction problem with a given current, I, the resistance can also be expressed as the electrical energy dissipation divided by the current squared

$$R_{\mathrm{e}} = \frac{\dot{E}_{\mathrm{ve}\phi}}{I^2} \qquad (23b)$$

As with electrical conduction, the entransy dissipation per unit time for one-dimensional heat conduction can be related to the thermal resistance as

$$\dot{E}_{\mathrm{vh}\phi} = \int_V \phi_{\mathrm{h}} \,\mathrm{d}V = \dot{Q}_{\mathrm{h}}^2 R_{\mathrm{h}} = \frac{(\Delta T)^2}{R_{\mathrm{h}}} \qquad (24a)$$

where

$$R_{\mathrm{h}} = \frac{(\Delta T)^2}{\dot{E}_{\mathrm{vh}\phi}} = \frac{\dot{E}_{\mathrm{vh}\phi}}{\dot{Q}_{\mathrm{h}}^2} \qquad (24b)$$

where $\dot{Q}_{\mathrm{h}}$ is the heat flow at the boundary. For a multi-dimensional domain with two iso- or non-isothermal boundary conditions, the equivalent thermal resistance can be written as

$$R_{\mathrm{h}} = \frac{(\Delta T)^2}{\dot{E}_{\mathrm{vh}\phi}} \quad \text{or} \quad R_{\mathrm{h}} = \frac{(\overline{\Delta T})^2}{\dot{E}_{\mathrm{vh}\phi}} \qquad (25a)$$

where $\overline{\Delta T}$ is the mean temperature difference. The equivalent thermal resistance is the ratio of the temperature difference squared over the entransy dissipation. For multi-dimensional heat conduction problems with specified heat flux boundary condition, the equivalent thermal resistance can be expressed as

$$R_{\mathrm{h}} = \frac{\dot{E}_{\mathrm{vh}\phi}}{\dot{Q}_{\mathrm{h}}^2} \qquad (25b)$$

The physics behind the entransy dissipation extremum principle can be understood from the relation between the entransy dissipation and the equivalent thermal resistance. According to Eq. (25b), the minimum entransy dissipation results in the minimum equivalent thermal resistance when the heat flux is prescribed. The maximum entransy dissipation corresponds to the minimum equivalent thermal resistance for prescribed boundary temperature according to Eq. (25a). Thus these two principles

can be combined into the minimum thermal resistance principle which states that for heat conduction problems with specified constraints (for instance, a limited amount of high conducting material in the substrate) the heat conduction is maximized or the heat conduction is optimized (maximum heat flux for a given temperature difference or the minimum temperature difference for a given heat flux) if the equivalent thermal resistance for the domain is minimized.

Therefore, optimization of a heat conduction process minimizes the equivalent thermal resistance for the constraints. The minimum entransy dissipation implies the minimum thermal resistance for a given boundary heat flux, while the maximum entransy dissipation implies the minimum thermal resistance for a given set of boundary temperatures.

5.3. Application of the entransy dissipation extremum principle (minimum thermal resistance principle)

5.3.1. Application to heat conduction

The volume-to-point conduction problem seeks to effectively remove heat generated in a volume to a point on its surface. High conductivity material is embedded in the substrate to improve the thermal conduction. The problem is to optimize the allocation of a limited amount of high conductivity material so that the generated heat can be most effectively transported to the point to minimize the temperature rise in the domain. Bejan [6,7] developed the constructal theory network of conducting paths to optimize the high conductivity material allocation so as to minimize the thermal resistance from the 'volume' to the 'point'. The material allocation in the volume-to-point conduction problem can also be optimized using the entransy dissipation concept.

Consider a square domain (two-dimensional) with a uniform heat source $\dot{q}_{hs} = 100 \text{ W/m}^3$, as shown in Fig. 6. The boundary has a constant temperature section at $T = 300 \text{ K}$. The heat generated in the domain is conducted to this small section with the other boundaries adiabatic. The local conductivity in the domain may vary continuously but the volume-averaged thermal conductivity is kept
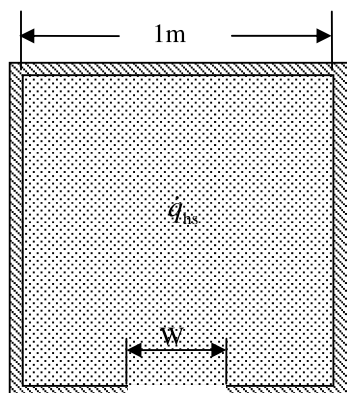

Fig. 6. Volume-to-point conduction problem.

constant at $\bar{k} = 1 \text{ W/m K}$. The objective is to optimize the conductivity distribution to minimize the domain temperature. The optimization objective is then the conductivity distribution in the domain that minimizes the entransy dissipation (gives the lowest average temperature of the domain). The constraint is that the volume-averaged conductivity is constant, i.e., $\int_V k \, dV = \text{const} = \bar{k} \cdot V$. This is a typical functional extremum problem.

The constraint is removed by using the Lagrange multipliers method to construct a functional,

$$J = \int_V \left[\frac{1}{2} k (\nabla T)^2 + \lambda k \right] dV \tag{26}$$

where λ is a Lagrange multiplier, which is a constant. By making the variation of the functional, J, with respect to the thermal conductivity, k, equal to zero, then

$$(\nabla T)^2 = 2\lambda \tag{27}$$

This equation demonstrates that the temperature gradient in the domain should be uniform to minimize the entransy dissipation. Using Fourier's law, Eq. (27) can also be written that the ratio of the local heat flux to the local thermal conductivity should be uniform over the domain to minimize the entransy dissipation.

The numerical procedure for finding the optimum distribution of high conductivity material that satisfies Eq. (27) is as follows:

(1) initially fill the domain with a uniformly distributed conductivity,
(2) solve the differential conduction equation to obtain the temperature field and heat flux field,
(3) calculate the new thermal conductivity distribution using the following equation:

$$k_{n+1}(x,y,z) = \frac{|\dot{q}_n(x,y,z)|}{\overline{|\dot{q}_n(x,y,z)|}} \cdot \overline{k_n(x,y,z)} \tag{28}$$

where $\dot{q}$ is the heat flux, n the iteration step, $\overline{|\dot{q}_n(x,y,z)|} = \frac{1}{V} \int_V |\dot{q}_n(x,y,z)| \, dV$, and $\overline{k_n(x,y,z)} = \frac{1}{V} \int_V k_n(x,y,z) \, dV = 1 \text{ W/m k}$,
(4) return to step (2) to recalculate the temperature and heat flux fields again until the following converging criterion is satisfied at each location,

$$(k_{n+1}(x,y,z) - k_n(x,y,z))/k_{n+1}(x,y,z) < \varepsilon \tag{29}$$

where ε is a positive value that is much less than one.

The numerically optimized thermal conductivity distribution is illustrated in Fig. 7. The constant temperature boundary section has two peaks that correspond to the heat flux peaks at each end. The peak value of the thermal conductivity is almost 200 times the average value, while in most of the domain the thermal conductivity is very small, only several percent of the average thermal conductivity. The temperature field before optimization (uniform thermal conductivity) is shown in Fig. 8, where the average temperature is 390.2 K, the highest temperature at the
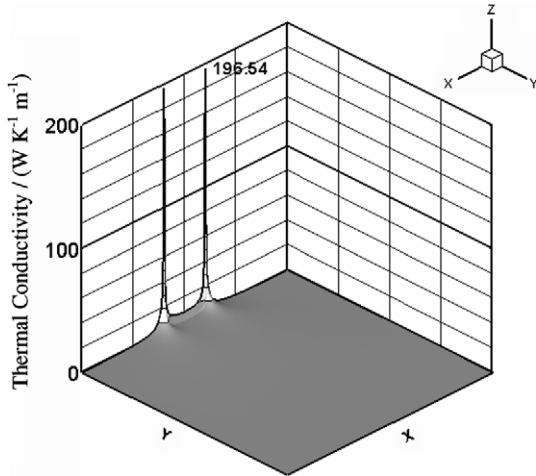
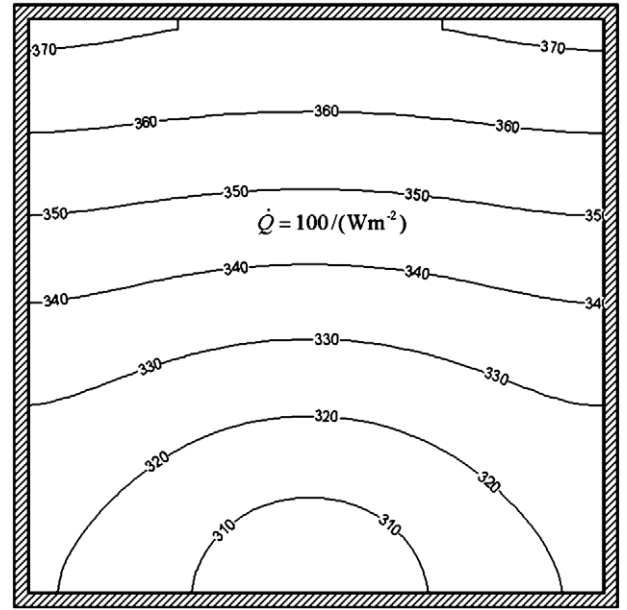Fig. 7. Thermal conductivity distribution after optimization for $W = 10$ cm.

Fig. 9. Temperature distribution after optimization for $W = 10$ cm.
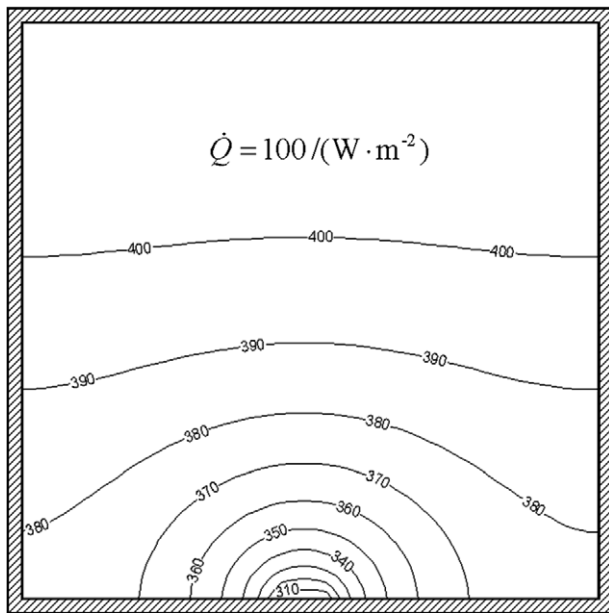

Fig. 8. Temperature distribution with uniform thermal conductivity distribution for $W = 10$ cm.

Fig. 10. Thermal conductivity distribution after optimization for $W = 2$ cm.

upper corner is 407.6 K, and the equivalent thermal resistance $R_h = \dot{E}_{vh\phi}/\dot{Q}_h^2 = 0.9$ K/W. The temperature gradients are more uniform after optimization, as shown in Fig. 9, where the average temperature is reduced from 390.2 K to 337.8 K and the highest temperature is reduced from 407.6 K to 372.2 K. The thermal resistance $R_h = \dot{E}_{vh\phi}/\dot{Q}_h^2 = 0.38$ K/W, which shows that the thermal resistance is reduced by more than half. When the isothermal section width is reduced from $W = 10$ cm to 2 cm there is only one thermal conductivity peak at the patch, as shown in Fig. 10, with a thermal conductivity of about twice that of the case with $W = 10$ cm. The equivalent thermal resistance is increased correspondingly.

For the case when a finite amount of highly conductive material is embedded in the substrate, the numerical procedure for finding the optimum allocation of high thermal conductivity material that approximately satisfies Eq. (23) is as follows:
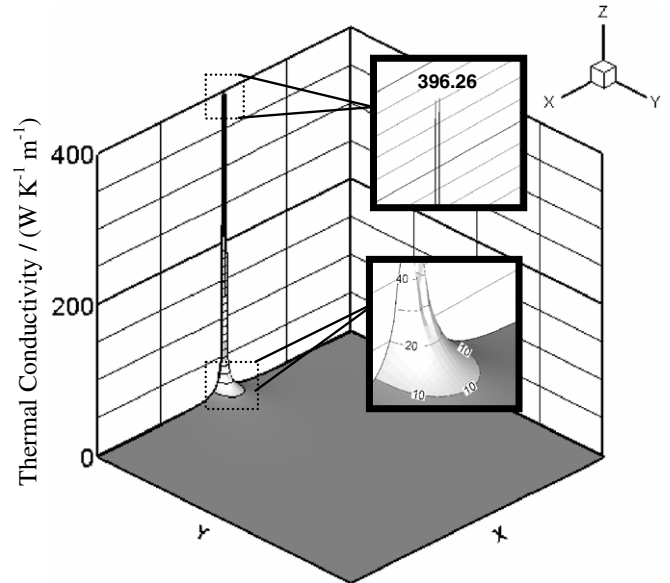
(1) The finite amount of high thermal conductivity material is divided into N portions

$$N = n\phi \tag{30}$$

where ϕ is the substrate porosity and n is the number of elements in the numerical model.

(2) Initially fill the domain with the low thermal conductivity substrate material.

(3) Solve the differential conduction equation to obtain the temperature field and heat flux field.

(4) The original substrate material is replaced by a portion of the high thermal conductivity material where such replacements maximize the entransy dissipation reduction.

(5) Return to step (3) to recalculate the temperature and heat flux field again and to step (4) to replace additional substrate material by another portion of the high thermal conductivity material, until all N portions of the high thermal conductivity material are embedded in the substrate.

Consider the volume-to-point conduction problem with a uniform heat source $\dot{q}_{hs} = 1000 \text{ W/m}^3$, the substrate material thermal conductivity, k_0, being 1 W/m K and the thermal conductivity of the additional material, k_P, being 400 W/m K, for the geometry shown in Fig. 6. The average temperature in the domain is 986 K while the highest temperature is 1127.3 K when the high thermal conductivity material is uniformly distributed. Simulations with three different mesh sizes were conducted to identify the mesh size effect. The calculated high conductivity materials distribution are illustrated in Fig. 11. The results for the thermal resistance R_h and the average temperature decrease are given in Table 2. The average temperature decrease for the finest mesh is 668.8 K, while the equivalent thermal resistance is $R_h = 0.0178$ K/W. Note that the differences in the average temperature decrease and the thermal resistance from the medium mesh to the finest mesh are very small. For comparison, the optimized results (second order assembly construct) using Bejan's constructal theory network of conducting paths are illustrated in Fig. 12. The
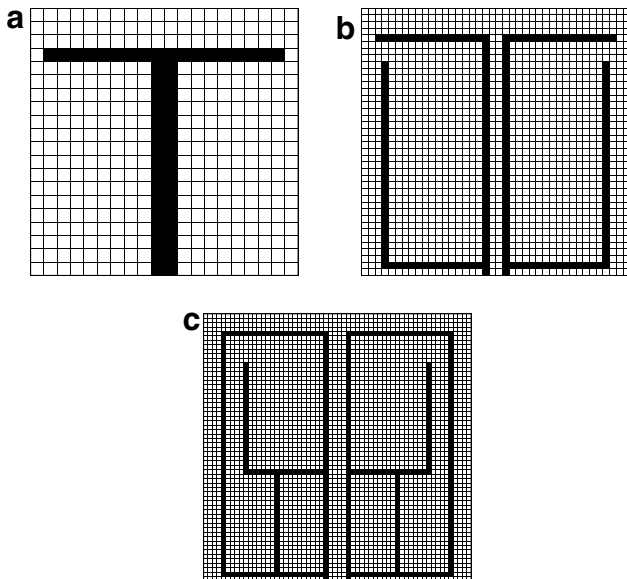
Table 2
Heat transfer results for the optimization of a finite amount of high thermal conductivity material in a uniform heat source material

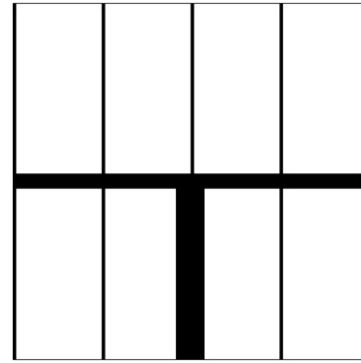| Mesh size	R_h (K/W)	Temperature decrease
20×20	0.0437	642.8
40×40	0.0179	668.7
60×60	0.0178	668.8

Fig. 12. Second order assembly construct using Bejan's constructal theory. Temperature decrease: 667.6 K.

average temperature decrease is 667.8 K and the equivalent thermal resistance $R_h = 0.0189$ K/W. Therefore, the average temperature decrease using the minimum thermal resistance principle based on the entransy dissipation concept for the finest mesh is slightly better than that predicted by using Bejan's secondary order assembly construct.

5.3.2. Application to heat convection

The extremum principle of entransy dissipation can also be applied to the optimization of heat convection processes. Consider heat convection in a straight tube with prescribed wall temperature and inlet fluid temperature. The objective is to find the optimal velocity distribution of the fluid in the tube for a fixed mass flow rate and flow resistance as constraints.

Meng et al. [16,17] used the maximum entransy dissipation principle to establish the following Lagrange function using variation calculus:

$$J = \int_V \left\{ \frac{1}{2} k (\nabla T)^2 + C_0 \phi + A_1 \left(k \nabla^2 T - \rho c_p U \cdot \nabla T \right) + C_1 \nabla \cdot U \right\} dV \tag{31}$$

where ϕ is the viscous dissipation function and C_0, A_1, C_1 are Lagrange multipliers. C_0 is required to be constant, while A_1 and C_1 are functions of U, T and position. The variation of J with respect to the temperature T yields

$$k \nabla^2 A_1 + \rho c_p U \cdot \nabla A_1 - k \nabla^2 T = 0 \tag{32}$$

and

$$\int_\Gamma \left\{ \left[k \nabla T - \left(k \nabla A_1 + \rho c_p U A_1 \right) \right] \delta T + A_1 \delta \nabla T \right\} d\vec{S} = 0 \tag{33}$$

Fig. 11. (a) Construct using the minimum thermal resistance principle (number of elements: 20×20). Temperature decrease: 642.8 K. (b) Construct using the minimum thermal resistance principle (number of elements: 40×40). Temperature decrease: 668.7 K. (c) Construct for the finest mesh using the minimum thermal resistance principle (number of elements: 60×60). Temperature decrease: 668.8 K.

The variation of J with respect to the velocity U gives

$$2C_0\mu\nabla^2U + \rho c_p U\nabla A_1 + \nabla C_1 = 0 \tag{34}$$

and

$$\int_\Gamma (2C_0P + C_1)\delta U \cdot \mathrm{d}\vec{S} = 0 \tag{35}$$

where μ is the dynamic viscosity and P is the pressure. Eqs. (32)–(35) can be combined to give

$$\mu\nabla^2U - \rho U \cdot \nabla U - \nabla P + (C_\phi A_1\nabla T + \rho U \cdot \nabla U) = 0 \tag{36}$$

where C_ϕ is a constant related to the given flow resistance (the given viscous dissipation). This is Euler's equation, which the fluid velocity field should satisfy to maximize the heat transfer and the resulting entransy dissipation in
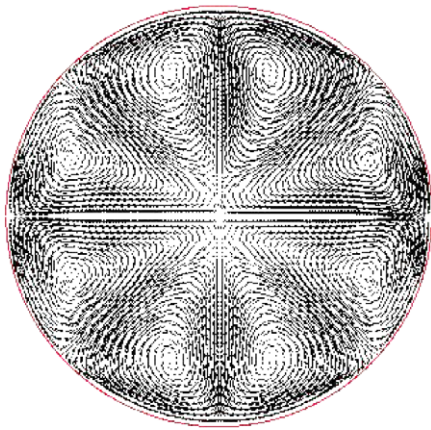
the tube flow. This equation was originally referred to as the field synergy equation with the entransy referred to as the heat transfer potential capacity [16].

Meng et al. [16] solved Eq. (36) numerically for fluid flow in a straight 20 mm diameter circular tube for fully developed laminar flow, a uniform wall temperature of 310 K and an average inlet fluid temperature is 300 K. A typical numerical result (at $Re = 400$ and $C_\phi = -0.01$) for the flow field is plotted in Fig. 13, where multiple longitudinal vortexes appear in the cross section of the circular tube. The Nusselt number is 313% higher than for fully developed laminar heat transfer in a circular tube. This implies that the heat convection in a tube can be optimized by generating multiple longitudinal vortices in the flow. Meng et al. [16,17] then developed a discrete double-inclined ribs tube and an alternating elliptical axis tube to produce the multiple longitudinal vortex flow. The numerically predicted flow field in the discrete double-inclined ribs tube is illustrated in Fig. 14, which is close to the

Fig. 13. Optimum flow field for laminar heat transfer in a circular tube ($Re = 400$).
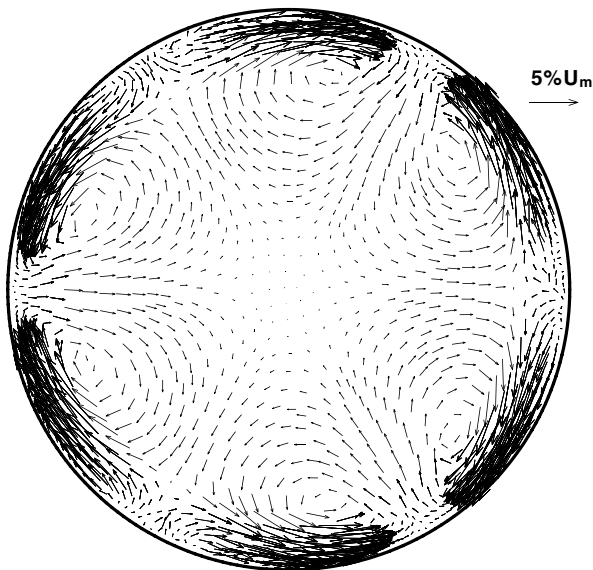
Fig. 14. Transverse flow velocities in the discrete double-inclined ribs tube.
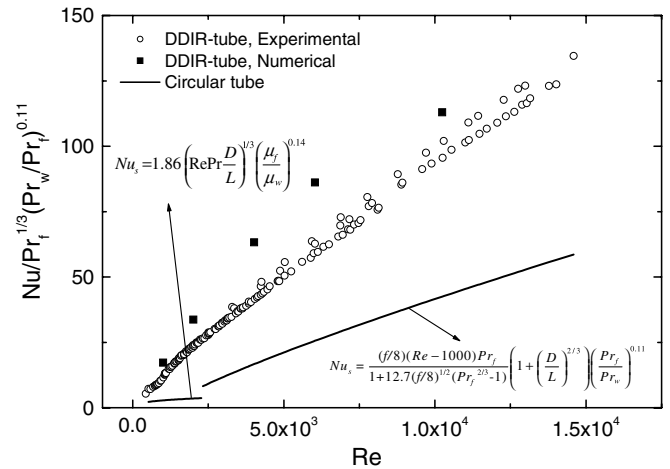

Fig. 15. Experimental and numerical results for the heat transfer and flow resistance in the discrete double-inclined ribs tube.
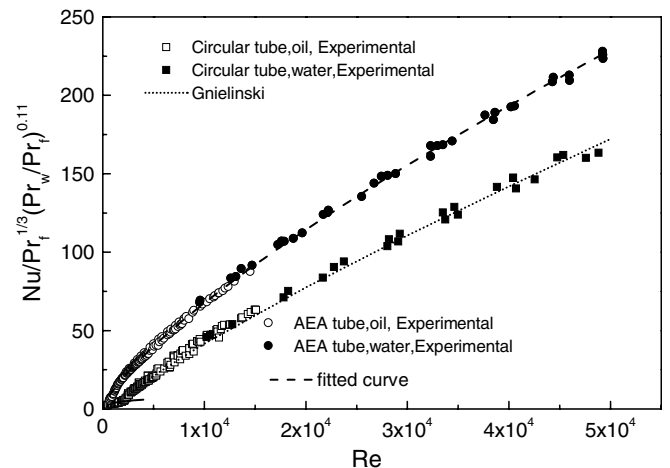

Fig. 16. Heat transfer in an alternating elliptical axis tube.

theoretically optimum flow field shown in Fig. 13. Both numerical and experimental results show that the enhanced tube has much better heat transfer performance than the smooth tube as shown in Fig. 15. Flow through an alternating elliptical axis tube also generates a multiple longitudinal vortex flow pattern, as shown in Fig. 14, which had dramatically enhanced heat transfer, as shown in Fig. 16.

6. Concluding remarks

(1) A new physical quantity, entransy, is introduced based on the analogy between heat conduction and electric conduction that can be used to optimize heat transfer processes. Without the concept of entransy heat transfer processes cannot be easily optimized. An analysis of the heat transfer from an object shows that the entransy, which corresponds to the electric potential energy in a capacitor, possesses the nature of "energy". Objects can then be described as thermal capacitors, which simultaneously store heat and "thermal potential energy".

(2) The entransy of an object is a measure of its ability to transfer heat to other object, just as the electrical energy of a capacitor is a measure of its ability to transfer charge. Heat transfer is always accompanied by entransy transfer. However, while thermal energy is conserved, entransy is not conserved due to dissipation. The entransy dissipation concept can then be used to define an efficiency for heat transfer processes and to optimize heat transfer processes.

(3) The extremum principle of entransy dissipation, which is equivalent to the minimum thermal resistance principle for heat conduction optimization, is derived using the method of weighted residuals.

(4) The entransy dissipation extremum principle or the minimum thermal resistance principle is applied to optimize the volume-to-point access thermal conduction problem. With a fixed volume-averaged conductivity as the constraint, the optimized thermal conductivity distribution is obtained that greatly reduces the equivalent thermal resistance and the average temperature.

(5) The entransy dissipation extremum principle is also applied to optimize heat convection of tube flow. With the mass flow rate and flow resistance as constraints, the optimum flow field in the tube is obtained which markedly maximize the heat transfer rate in the tube for a given temperature boundary condition.

References

[1] L. Onsager, Reciprocal relations in irreversible processes. II, Phys. Rev. 38 (12) (1931) 2265–2279.
[2] L. Onsager, S. Machlup, Fluctuations and irreversible processes, Phys. Rev. 91 (6) (1953) 1505–1512.
[3] I. Prigogine, Introduction to Thermodynamics of Irreversible Processes, third ed., Wiley, New York, 1967, pp. 76–77.
[4] A. Bejan, Entropy Generation through Heat and Fluid Flow, Wiley, New York, 1982, pp. 118–134.
[5] A. Bejan, Entropy Generation Minimization, CRC Press, Florida, 1996, pp. 47–112.
[6] A. Bejan, Constructal-theory network of conducting paths for cooling a heat generation volume, Int. J. Heat Mass Transfer 40 (4) (1997) 799–808.
[7] A. Bejan, Shape and Structure, from Engineering to Nature, Cambridge University Press, New York, 2000, pp. 52–81.
[8] P.J. Schneider, Conduction Heat Transfer, Addison-Wesley, USA, 1955, pp. 338–339.
[9] Z.Y. Guo, X.G. Cheng, Z.Z. Xia, Least dissipation principle of heat transport potential capacity and its application in heat conduction optimization, Chin. Sci. Bull. 48 (4) (2003) 406–410 (in Chinese).
[10] M.A. Biot, Variational principle in irreversible thermodynamics with applications to viscoelasticity, Phys. Rev. 97 (6) (1955) 1463–1469.
[11] E.R.G. Eckert, R.M. Drake, Analysis of Heat and Mass Transfer, McGraw-Hill, New York, 1972, pp. 19–23.
[12] X.G. Cheng, Z.X. Li, Z.Y. Guo, Variational principles in heat conduction, J. Eng. Thermophys. 25 (3) (2004) 457–459 (in Chinese).
[13] X.G. Cheng, Entransy and its application in heat transfer optimization, Ph.D. thesis, Tsinghua University, Beijing, 2004 (in Chinese).
[14] B.A. Finlayson, L.E. Scriven, On the search for variational principles, Int. J. Heat Mass Transfer 10 (1967) 799–821.
[15] B.A. Finlayson, The Method of Weighted Residuals and Variational Principles, with Application in Fluid Mechanics, Heat and Mass Transfer, Academic Press, New York, 1972, pp. 347.
[16] J.A. Meng, X.G. Liang, Z.X. Li, Field synergy optimization and enhanced heat transfer by multi-longitudinal vortexes flow in tube, Int. J. Heat Mass Transfer 48 (16) (2005) 3331–3337.
[17] J.A. Meng, X.G. Liang, Z.X. Li, Z.Y. Guo, Numerical study on low Reynolds number convection in alternate elliptical axis tube, J. Enhanced Heat Transfer 11 (4) (2004) 307–313.